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A New Approach to the Inverse Frobenius-Perron Problem for Prime Gaps: Logistic-Based Chaotic Maps

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ABSTRACT

Given the invariant density or ergodic distribution, we can find some dynamical map $f(.)$ such that $x_{n+1} = f(x_n)$ for which the collection $\{x_n\}$ of prime gaps follow the stationary distribution referred to as the Inverse Frobenius-Perron map. We provide an alternative theory for finding such a formula based on chaotic dynamics.

Keywords: Prime gaps, invariant density, chaotic map, inverse Frobenius-perron

INTRODUCTION

A dynamical system $\{X_t\}$ generated by a map $\psi(.)$:

$$X_{t+1} = \psi(X_t), \quad t = 0,1,2,...N \tag{1}$$

is chaotic if the values $\{X_t\}$ behave like a random sequence. Devaney (2000) provides a more rigorous definition of chaos which essentially boils down to: (a.) sensitivity to initial conditions, (b.) topological transitivity, and (c.) countably infinite periodic points of all periods. By treating (1) as a pseudo-random sequence for large N, one obtains a probability distribution $F(.)$, assumed absolutely continuous with respect to a Lebesgue measure, that describes the random behavior of the sequence. The invariant distribution *F*(.) is the fixed point of a Frobenius-Perron operator *L*(.) (Pingel, 1989):

$$\rho_{n+1}(x) = L_{\psi}(\rho_n(x))$$

where $\rho_n\ (x)$ is the density at the nth iterate. In the case of the logistic map:

$$X_{t+1} = \psi\ (X_t) = 4X_t(1-X_t), \quad t = 0,1,2, \dots N \tag{2}$$

The Frobenius-Perron invariant distribution is analytically derived as:

$$F(x) = \frac{2\text{arcsin}\ (\sqrt{x}}{\pi}\ , \quad 0 < x < 1 \tag{3}$$

$$f(x) = \frac{1}{\pi\sqrt{x(1-x)}} \tag{4}$$

In many practical situations, the invariant distribution $F(.)$ is known and the problem is to find the chaotic map $\psi(.)$ that generated the values. This is the inverse Frobenius-Perron problem (Diakonos, 1997; Nijun-Wei , 2013). Nijun-Wei (2013), Pingel (1989) and others have enumerated four different approaches to the problem: (a.) method of conjugation, (b.) differential equation approach, (c.) Pingel's approach and (d.) matrix approach. In this Chapter, we introduce an approach that is based on the dynamics of a known chaotic map ψ (.) and a known invariant distribution F(.) to construct an unknown chaotic map $\theta(.)$ whose invariant distribution $H(.)$ is known.

The approach uses the inverse transform theorem which states that if *F*(*x*) is the distribution of a random variable *X*, then:

$$U = F(x) \underset{\sim}{d}\ U(0,1)$$

is uniformly distributed on (0,1). Hence, if $H(y)$ is the distribution of another random variable Y, then

$$U = F(x) = H(y) \underset{\sim}{d}\ U(0,1) \tag{5}$$

Equation (5.5) then allows us to connect the dynamics of Y with the dynamics of X. That is,

$$H(Y_t\ ) = F(X_t) = U_t \underset{\sim}{d}\ U(0,1) \text{ for each } t \tag{6}$$

and so,

$$Y_t = H^{-1}(F(X_t)) \tag{7}$$

Chaotic Maps Derived by Conjugation of an Auxillary Map

Let $I = \{x : x \in [0,1]\}$ and $\psi : I \to I$ be given by Equation (2). Let $J = \{y : y \in (0,\infty)\}$ and $\theta : J \to J$ be an unknown chaotic map:

$$Y_{t+1} = \theta(Y_t)$$

The invariant distribution of (2) is given by Equation (3) while the invariant distribution of $\{Y_t\}$ is a known distribution $H(y)$.

Theorem 1: Let $\psi : I \to I$ be a chaotic map with invariant distribution $F(.)$ and let $\theta : J \to J$ be another chaotic map with invariant distribution $H(.)$. Then:

$$\theta = (F^{-1} * H)^{-1} * \psi * (F^{-1} * H)$$

Note that if **G =** $\boldsymbol{F^{-1} * H}$, then: $\boldsymbol{\theta = G^{-1} * \psi * G}$.

Proof. From (6) and (7):

$$Y_t = H^{-1}\big(F(X_t)\big) = H^{-1}\big(F(\psi(X_{t-1})\big) = H^{-1}\big(F(\psi(F^{-1}(U_{t-1})\big)$$
$$= \big(F(\psi(F^{-1}(H(Y_{t-1})\big) \blacksquare$$

By Theorem 1, $\theta(.)$ is obtained by conjugation of $\psi(.)$. The map $\theta(.)$ inherits the dynamical characteristics of $\psi(.)$. This is not the same as the conjugation approach of Nijun-Wei (2015) who derived $\theta(.)$ independently of $\psi(.)$.

Let $H(.)$ be the exponential distribution:

$$H(y) = 1 - \exp(-\lambda y) \text{ , } \lambda > 0, y > 0 \qquad (8)$$

where $\lambda \approx \frac{1}{\log(N)}$ for large N. Equation (8) arises as an approximation to the distribution of prime gaps:

$$Y_n = P_{n+1} - P_n \text{ , } n = 1,2,3, \dots N$$

where P_{n+1} and P_n are consecutive primes (Cramer, 1936; Selberg, 1948; Yamasaki and Yamasaki, 1991). On the dynamics of prime gaps (Libao 2016 dissertation), we established that the gaps $\{Y_n\}$ form a chaotic sequence with a periodic point of period 3. Li and Yorke (1975) demonstrated that if a system has a period 3 point, then the system is chaotic.

From (5) and (8), we have:

$$\frac{2\arcsin(\sqrt{X_t}}{\pi} = 1 - \exp(-\lambda Y_t)$$

Hence,

$$Y_t = \frac{-1}{\lambda}\ln(1 - \frac{2}{\pi}\arcsin\sqrt{\psi(X_{t-1})})$$

However,

$$X_{t-1} = sin^2(\frac{\pi}{2}(1 - \exp(-\lambda Y_{t-1})).$$

Let $\varphi_{t-1} = \frac{\pi}{2}(1 - \exp(-\lambda Y_{t-1}))$, then:

$$Y_t = \frac{-1}{\lambda}\ln(1 - \frac{2}{\pi}\arcsin\sqrt{4X_{t-1}(1 - X_{t-1})}) \tag{9}$$

$$Y_t = \frac{-1}{\lambda}\ln(1 - \frac{4}{\pi}\varphi_{t-1})$$

Equation (9) further simplifies to:

$$Y_t = \frac{-1}{\lambda}\ln|-1 + 2\exp(-\lambda Y_{t-1})|$$

Hence,

$$\theta(y) = \frac{-1}{\lambda}\ln| -1 + 2\exp(-\lambda y)| .$$

Theorem 2. Let $\psi(x) = 4x(1 - x)$ with invariant distribution:

$$f(x) = \frac{1}{\pi\sqrt{x(1-x)}}, 0 < x < 1$$

and let $\theta(y)$ be the dynamical map for the prime gaps with invariant distribution:

$$h(y) = \lambda\exp(-\lambda y), \qquad \lambda \approx \frac{1}{\log(N)},$$

then,

$$\theta(y) = \frac{-1}{\lambda}\ln| -1 + 2\exp(-\lambda y)| .$$

The dynamical map θ mimics the random pattern generated by the logistic map. In fact:

Corollary 1. The fixed points $X_0 = 0\ and\ X_{0'} = \frac{3}{4}$ of the logistic map are mapped onto the fixed points $Y_0 = 0\ and\ Y_{0'} = \frac{\ln(3)}{\lambda}$ of the map θ.

Proof. We show that the points $Y_0 = 0\ and\ Y_{0'} = \frac{\ln(3)}{\lambda}$ are fixed points of θ. For $Y_0 = 0$, we have:

$$\theta(0) = \frac{-1}{\lambda}\ln(-1 + 2\exp(-\lambda(0))) = \frac{-1}{\lambda}\ln(-1 + 2(1)) = \frac{-1}{\lambda}(0) = 0.$$

For the second point, we note that $-1 + 2\exp(-\lambda y) < 0$, , hence:

$$\frac{-1}{\lambda}\ln[-(-1 + 2\exp(-\lambda y))] = y$$

It follows that:

$$1 - 2\exp(-\lambda y) = \exp(-\lambda y)$$

$$\exp(-\lambda y) = \frac{1}{3}$$

$$y = \frac{\ln(3)}{\lambda}$$

$$and\ \theta\left(\frac{\ln(3)}{\lambda}\right) = \frac{\ln(3)}{\lambda} \quad \blacksquare$$

Figure 1 shows the plot of $\theta(y)$ against y for $N = 100{,}000{,}000$.

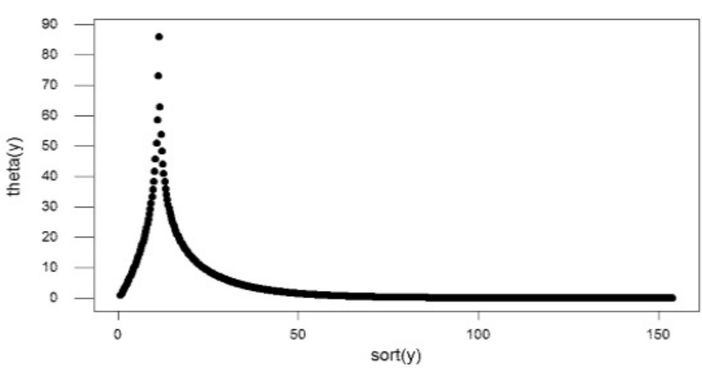

Figure 1. Theta (y) versus y.

Figure 2 shows the phase diagram of $Y(t+1)$ versus $Y(t)$ using the initial condition $Y_0 = .06$.

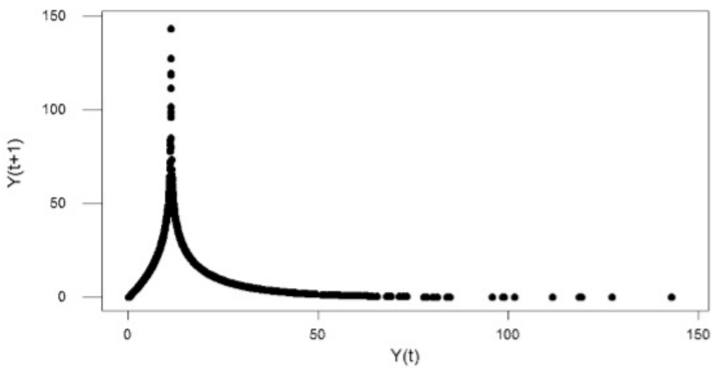

Figure 2. Phase diagram of $Y(t+1)$ versus $Y(t)$.

On the other hand, we also viewed the phase diagram for $E(Y_{t+1}|Y_t=y)$ versus $Y_t=y$ for the actual prime gaps up to gap = 18 as shown in Figure 3:

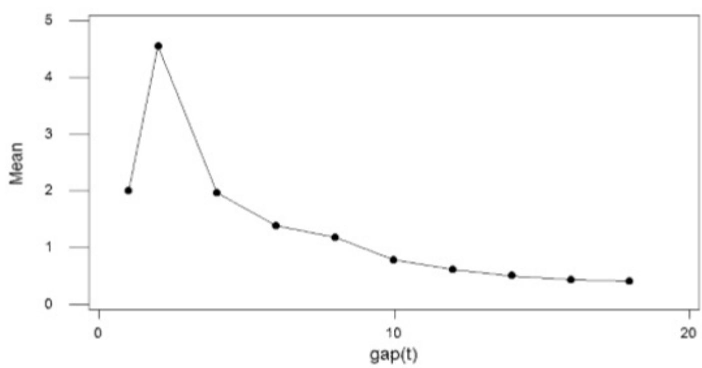

Figure 3. Phase diagram of $E(Yt+1|Yt)$ and Yt.

We verify that the pseudo-random numbers generated by $\theta(Y_{t-1})$ follow an exponential distribution. This is contained in Theorem 3.

Theorem 3. Let $X_{t+1} = 4X_t(1-X_t)$ with ergodic distribution $Beta\left(\frac{1}{2},\frac{1}{2}\right)$. Then:

$$Y_t = -\frac{1}{\lambda}\ln\left|1-\frac{2arcsin(\sqrt{x_t})}{\pi}\right|$$

has an exponential distribution with rate parameter λ.

Proof: For each t,

$$F(x_t) = \frac{2[arcsin(\sqrt{x_t})]}{\pi} = u_t = 1 - e^{-\lambda y_t} \text{ where } u_t \underset{\sim}{d} U(0,1).$$

The result follows. ■

Theorem 3 suggests the following algorithm:

Algorithm:

1. Choose an initial value $X_0 \in [0, 1]$ away from a fixed point of the logistic map;
2. Compute $X_{t+1} = 4X_t(1 - X_t)$, $t = 0, 1, 2, \ldots, n$;
3. Compute $Y_t = -\frac{1}{\lambda}\ln\left|1 - \frac{2arcsin(\sqrt{x_t})}{\pi}\right|$, $t = 0, 1, 2, \ldots, n$;
4. Return Y_t

The specific prime gaps sequence {2, 2, 4, 2, 4, 2, 4, 6, …} is just one of the infinitely many possible paths that $\{Y_t\}$ can take. Note, however, that $X_t = \psi(X_{t-1})$ and given step 3 of the algorithm, the behavior of $\{Y_t\}$ is completely determined by the behavior of $\psi(.)$. Likewise, given the dynamical character of $\{X_t\}$, the initial value X_0 completely specifies the trajectory of the system.

Theoretical Bound for the Mean Absolute Error

Consider:

$$\frac{1}{n}\sum_{t=0}^{n} |(Y_t - Y_{t'})|$$

representing the average divergence of the two trajectories of the chaotic paths $\{Y_t\}$ and $\{Y_{t'}\}$. The **Lyapunov characteristic exponent** of a dynamical system is a quantity that characterizes the rate of separation of infinitesimally close trajectories (Bryant *et al.*, 1990). Two trajectories diverge at a rate given by:

$$d(y) \approx \exp(\delta(y))\,|d(y_0)| = k\exp(\delta(y))$$

where $d(y_0)$ is the initial separation of the trajectories. The quantity $\delta(y)$ is the Lyapunov exponent. This is given by:

$$\delta(y) = \lim_{n\to\infty}\frac{1}{n}\sum_{n=0}^{\infty}\ln(|\theta'(y)|)$$

Since the invariant distribution of $\{Y_t\}$ is known, we can compute:

$$\delta(y) = \int_0^{\infty} ln|\theta'(y)|\,dH(y)$$

It is easy to see that:

$$\theta'(y) = \frac{-2}{\exp(\lambda y)-2}$$

And

$$|\theta'(y)| = \frac{2}{\exp(\lambda y) - 2} \ \ for\ large\ y$$

Hence,

$$\delta(y) = \int_0^\infty ln|\theta'(y)|\, dH(y) = \ln(2) - \int_1^\infty \frac{\ln|u-2|}{u^2}\, du\ ,\ u = \exp(\lambda y)$$

The integral on the right converges to

$$\delta(y) = \int_0^\infty ln|\theta'(y)|\, dH(y) = \ln(2) - \int_1^\infty \frac{\ln|u-2|}{u^2}\, du$$
$$\approx \ln(2) - \left[\int_3^\infty \frac{-1}{2u}\, du + \int_3^\infty \frac{1}{2(u-2)} du\right]$$
$$\approx \ln(2) + \frac{1}{2}\int_3^\infty \frac{1}{u}\, du - \frac{1}{2}\int_3^\infty \frac{du}{(u-2)}$$
$$\approx \ln(2) + \frac{1}{2}\ln\left(\frac{u}{u-2}\right)_3^\infty = \ln(2) - \frac{1}{2}\ln(3)$$

when the singularity at $u = 2$ is avoided. In this case, the Lyapunov exponent is roughly $\delta = 0.143841$. Consequently,

$$\frac{1}{n}\sum_{t=0}^{n} |(Y_t - Y_{t'})| \sim \mathrm{k}\exp(\delta)\log(n) = 1.15470\, k\log(n)\ where\ |k| \leq 1$$

The Lyapunov exponent for the chaotic logistic map is $\ln(2)$ (Pesin, 1977). We observe that the Lyapunov exponent obtained for the chaotic map $\theta(y)$ is less than $\ln(2)$ indicating that $\theta(y)$ is less chaotic than $\psi(x)$.

The smallest prime gap is $y = 2$ and for this $u = \exp(\lambda y) = 1.11469$. The Cauchy principal value of the full integral is:

$$-\int_1^\infty \frac{\ln|u-2|}{u^2}\, du = \frac{1}{2}\ln\left(\frac{u}{|u-2|}\right)_1^\infty = 0$$

and:

$$\delta(y) = \int_0^\infty ln|\theta'(y)|\, dH(y) = \ln(2) - \int_1^\infty \frac{\ln|u-2|}{u^2}\, du = \ln(2)\ , \qquad u = \exp(\lambda y)$$

from which we recover the Lyapunov exponent of the chaotic logistic map. It follows that:

$$\frac{1}{n}\sum_{t=0}^{n} |(Y_t - Y_{t'})| \sim k\exp(\delta)\log(n) = 2k\log(n)\, ,\, |k| \leq 1 \qquad (10)$$

Theoretical Absolute Error

Next, we consider the magnitude of the difference between the *n*th prime gap (*Yn*) and the nth prime gap prediction (*Yn'*):

$$\|Y_n - Y_{n'}\| \leq \left|\log(n) - \ln\left(\frac{2}{\pi}\arcsin\left(\sqrt{.5000}\right)\right)\ln(n)\right|$$

$$\|Y_n - Y_{n'}\| \leq |1 - 0.693147|\ln(n) = 0.306853\ln(n)$$

where $X_0 = 0.50000 = E(X)$ of the arcsine distribution (4).

Theorem 4. Let Y_n be the nth prime gap and let $Y_{n'}$ be the estimated nth prime gap based on the chaotic map $\theta(y)$ obtained by conjugation of the chaotic logistic map $\psi(x)$. Then:

$$\|Y_n - Y_{n'}\| \leq |1 - 0.693147|\ln(n) = 0.306853\ln(n)$$

Numerical Simulation

Let {*Yt*} = {1,2,2,4,2,4,2,4,6,2,6,…} be the natural sequence of prime gaps starting from the first gap of 1. Using the exponential approximation to the distribution of prime gaps with:

$$f(y) = \lambda e^{-\lambda y}$$

and $\lambda \approx \frac{1}{\log(n)}$,

then,

$$x_t = sin^2\left(\frac{\pi}{2}\left(1 - e^{-\lambda y_t}\right)\right), n = 10^8.$$

We obtain the initial condition for the chaotic logistic map by substituting $y_0 = 1$ to obtain:

$$x_0 = 0.00687$$

where $\log(100{,}000{,}000) = 18.4207$. The sequence induced by the logistic map with this initial condition is given by:

$$x_{t+1} = 4x_t(1 - x_t), \quad x_0 = 0.00687$$

These will produce the sequence $\{y_t'\} = \{1, y_1' y_2', \ldots\}$ which has an exponential distribution with the same rate parameter λ as the original distribution of y_t.

The mean absolute error computed for this sequence is:

$$MAE = \frac{1}{1228}\sum_{t=0}^{1227}|Y_t - Y_t'| = 14.127$$

for the first 1,228 prime gaps.

We then proceeded to examine the behavior of the sequence in the neighborhood of x_0=0.00687. Table 1 shows the MAE obtained for various choices of initial values in the neighborhood of x_0.

Table 1. MAE for Various Initial Values Near X_0 = 0.00687

MAE	Initial Values X_0
14.467	0.00287
14.417	0.00387
14.320	0.00487
14.369	0.00587
14.126	0.00687
14.342	0.00700
14.303	0.00800
14.615	0.00900
14.268	0.01000
14.414	0.01500
14.716	0.02000
14.072	**0.02500**
14.311	0.03000
14.458	0.03300
14.186	0.03400
14.345	0.03500
14.608	0.03362

The minimum value of the MAE occurs at x_0 = 0.025 with MAE = 14.072. The graph of initial values versus MAE is shown below:

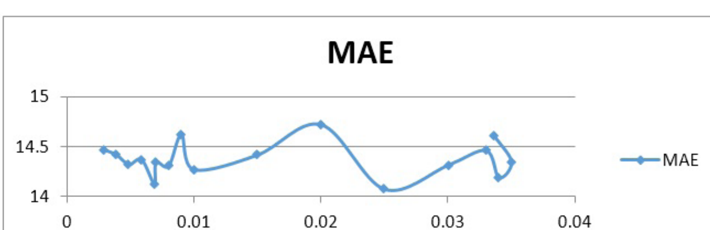

Figure 4. Mean Absolute Errors for Various Initial Values.

The histogram for the optimal sequence of gaps generated from the initial condition x = 0.025 is shown below to be exponential as desired:

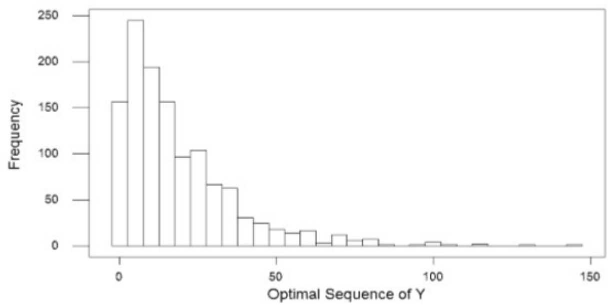

Figure 5. Histogram for the Optimal Sequence of Y Generated from X = 0.025.

Figure 4 shows that the objective function (MAE) for the minimization problem has several local minima within (.003,.035). However, we also note that the minimum among the local minima satisfies:

$$Min_{y_t'}(min\frac{1}{1228}\sum_{t=0}^{1227}|Y_t - Y_t'| \leq 0.763923\log(n) \approx \gamma\log(n)$$

where $\gamma = Ramanujan - Landau\ constant = .764223$ or using (10), $k = .661577$.

The initial condition used previously was based on the fixed point x = 0 and the initial prime gap of y = 1. We can also explore the other unstable fixed point at x = .75 and use the following relation

$$Y_t = -\frac{1}{\lambda}\ln\left|\frac{2arcsin(\sqrt{x_t})}{\pi}\right|$$

Table 2 shows the results of the exploration around the neighborhood of 0.75. We find that the objective function has a unique minimum at around X_0 = 0.739 and:

$$min\frac{1}{1228}\sum_{t=0}^{1227}|Y_t - Y_t'| \leq 0.7366712\log(n) \approx \gamma^{1.13655}\log(n) \text{ or using (10),}$$

$$k = 0.6370763$$

where y is the Ramanujan-Landau constant earlier given. Table 2 shows the results of the exploration.

Table 2. MAE for Initial Values Near an Unstable Fixed Point X_0=0.75

Initial Values	MAE
0.7300	15.941
0.7350	15.292
0.7370	14.476
0.7390	**13.570**
0.7400	13.729
0.7401	15.323
0.7420	15.532
0.7600	15.960

Figure 6 shows the mean absolute errors for various initial values used.

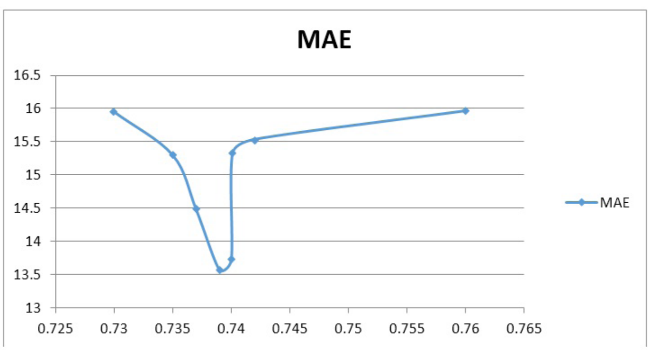

Figure 6. MAE around the neighborhood of x = 0.75.

Figure 7 shows the histogram of the optimal sequence of gaps generated with an initial condition of x_0 = 0.739.

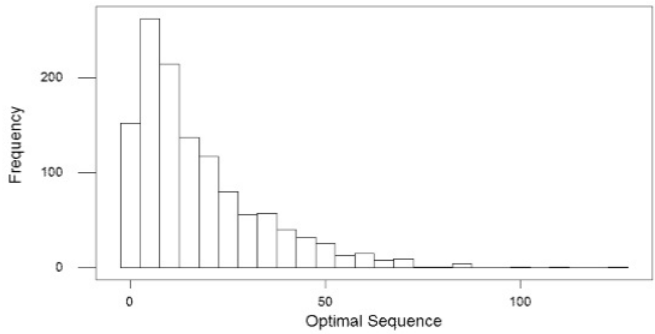

Figure 7. Histogram of the Optimal Sequence of Gaps with Initial Condition $X_0 = 0.739$

Absolute Errors

Table 3 shows the estimated nth prime $P_{n'}$ and the actual nth prime P_n with their absolute errors and relative absolute errors:

Table 3. Absolute Error for the Estimation of the nth Prime

N	P(N-1)	Actual P(N)	Estimated Gap	P(N)-Estimated	ABS(Error)	Relative Abs Error
5000	48593	48611	5.90361	48599	12	0.000246858
10000	104723	104729	6.38406	104729	0	0.000000000
15000	163819	163841	6.6651	163826	15	9.15522E-05
30000	350351	350377	7.14555	350358	19	5.42273E-05
100000	1299689	1299709	7.98007	1299697	12	9.23284E-06
200000	2750131	2750159	8.46052	2750139	20	7.27231E-06
300000	4256227	4256233	8.74156	4256236	3	7.04849E-07
400000	5800057	5800079	8.94097	5800066	13	2.24135E-06
500000	7368743	7368787	9.09563	7368752	35	4.74976E-06
600000	8960447	8960453	9.22201	8960456	3	3.34805E-07
664578	9999971	9999973	9.29286	9999980	7	7.00002E-07

Table 4 compares the estimated nth prime using the dynamical map $\theta(y)$ and the Prime Number Theorem prediction of the nth prime:

Table 4. PNT versus Chaotic Map Prediction of the nth Prime

N	Actual Nth Prime Number	Predicted P_N Using PNT	Chaotic Map Estimate of P_N
5000	48611	42586	48599
10000	104729	92103	104729
15000	163841	144237	163826
30000	350377	309269	350358
100000	1299709	1151293	1299697
200000	2750159	2441215	2750139
300000	4256233	3783461	4256236
400000	5800079	5159688	5800066
500000	7368787	6561182	7368752
600000	8960453	7982811	8960456
664578	9999973	8909936	9999980

Tabular values show that the predictions made using the chaotic dynamics approach are much closer to the actual primes than the predictions using the Prime Number Theorem.

CONCLUSION

With the concept of deterministic randomness, we modelled the prime gaps as a chaotic dynamical system. Given the invariant density or ergodic distribution, we can find some dynamical map f (.) such that: $x_{n+1} = f(x_n)$ for which the collection $\{x_n\}$ of prime gaps follow the stationary distribution referred to as the Inverse Frobenius-Perron map. This map is given by

$$\theta(y) = \frac{-1}{\lambda}\ln|\ -1 + 2\exp(-\lambda y)|\ .$$

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